

# ON EQUIVARIANT FORMAL DEFORMATION THEORY

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**ABSTRACT.** Using the set-up of deformation categories of Talpo and Vistoli, we re-interpret and generalize, in the context of cartesian morphisms in abstract categories, some results of Rim concerning obstructions against extensions of group actions in infinitesimal deformations. Furthermore, we observe that finite étale coverings can be infinitesimally extended and the resulting formal scheme is algebraizable. Finally, we show that pre-Tango structures survive under pullbacks with respect to finite, generically étale surjections  $\pi : X \rightarrow Y$ , and record some consequences regarding Kodaira vanishing in degree one.

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## INTRODUCTION

In deformation theory, one often seeks to extend automorphisms along infinitesimal extensions. This is not always possible: For example, Serre [14] showed that there are flat families of smooth hypersurfaces  $X \subset \mathbb{P}^4$  over  $\Lambda = \mathbb{Z}_p$  whose closed fiber  $X_0$  comes with a free action of some elementary abelian  $p$ -group  $G$  that does not extend to all infinitesimal neighborhoods  $X_n$ . Furthermore, the resulting quotient  $Y_0 = X_0/G$  then does not lift to characteristic zero.

Rim [12] developed a formalism that explains the *obstructions* in terms of certain group cohomology in degree one and two. Our motivation for this paper is to elucidate and perhaps simplify Rim's arguments by extending them into a purely categorical setting, merely using Grothendieck's notion of *cartesian morphisms* for functors  $p : \mathcal{F} \rightarrow \mathcal{E}$  between arbitrary categories [7], much in the spirit of Talpo and Vistoli [18].

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Recall that a morphism  $f : \xi \rightarrow \xi'$  in  $\mathcal{F}$  over a morphism  $S \rightarrow S'$  in  $\mathcal{E}$  is cartesian if, intuitively speaking,  $\xi$  behaves like a “base-change” of  $\xi'$  to  $S$ . Now let  $\xi \in \mathcal{F}$  be an object over some  $S \in \mathcal{E}$ , and  $G \rightarrow \text{Aut}_S(\xi)$  be a homomorphism of groups. Write  $\text{Lif}(\xi, S')$  for the set of isomorphism classes of cartesian morphisms  $\xi \rightarrow \xi'$  over  $S \rightarrow S'$ . This set is endowed with a  $G$ -action, by transport of structure. Fix a cartesian morphism  $f : \xi \rightarrow \xi'$ . Our first main result is the following:

**Theorem.** (See Theorem 1.2.) *In the above setting, suppose that the group  $\text{Aut}_\xi(\xi')$  is abelian. Then the  $G$ -action on  $\xi$  extends to a  $G$ -action on  $\xi'$  if and only if the following two conditions hold:*

- (i) *The isomorphism class  $[f] \in \text{Lif}(\xi, S')$  is fixed under the  $G$ -action.*
- (ii) *The resulting cohomology class  $[\tilde{G}] \in H^2(G, \text{Aut}_\xi(\xi'))$  is trivial.*

Here  $\tilde{G} = \text{Aut}_{S'}(\xi') \times_{\text{Aut}_S(\xi)} G$  is the induced extension of  $G$  by  $\text{Aut}_\xi(\xi')$ , and  $[\tilde{G}]$  is the resulting cohomology class.

We then apply this to the following algebro-geometric setting, using the set-up of Talpo and Vistoli [18]: Let  $\Lambda$  be a complete local noetherian ring, with residue field  $k = \Lambda/\mathfrak{m}_\Lambda$ , and  $\mathcal{F} \rightarrow (\text{Art}_\Lambda)^{\text{op}}$  be a *deformation category*, that is, a category fibered in groupoids that satisfies the *Rim–Schlessinger Condition*. The latter is a technical condition that comes from the structure theory of flat schemes over Artin rings. Note that the ring  $\Lambda$  may be of mixed characteristics, which was not allowed in [12]. Let  $\xi \in \mathcal{F}(A)$  be an object, and  $A' \rightarrow A$  be a small extension of rings, and  $\xi_0 = \xi|_k$ . We then use an observation of Serre from [15] and regard the set  $\text{Lif}(\xi, A')$ , if nonempty, as a *torsor with a group of operators*  $G$ , to get a cohomology class

$$(1) \quad [\text{Lif}(\xi, A')] \in H^1(G, I \otimes_k T_{\xi_0}(\mathcal{F})).$$

This class is trivial if and only if there is some extension  $\xi \rightarrow \xi'$  whose *isomorphism class* is  $G$ -fixed. The actual  $G$ -action on  $\xi$  extends to such an object  $\xi' \in \mathcal{F}(A')$  if and only if the ensuing cohomology class

$$(2) \quad [\tilde{G}] \in H^2(G, \text{Aut}_\xi(\xi')) = H^2(G, I \otimes_k \text{Aut}_{\xi_0}(\xi_{k[\epsilon]}))$$

vanishes. Summing up, we have a primary obstruction (1), which deals with  $G$ -actions on isomorphism classes, and a secondary obstruction (2), which takes care of the actual  $G$ -action on objects.

If  $G$  is finite and the residue field  $k = \Lambda/\mathfrak{m}_\Lambda$  has characteristic  $p > 0$ , then the above obstructions actually lie in the corresponding cohomology groups for a Sylow  $p$ -subgroup  $P \subset G$ . Consequently, the  $G$ -action extends if and only if the  $P$ -action extends.

We also take up two closely related topics: First, we verify that finite étale coverings can be infinitesimally extended and the resulting formal scheme is always *algebraizable*. Second, we show that *pre-Tango structures* survive under pullbacks with respect to finite, generically étale surjections  $f : X \rightarrow Y$ , and record some consequences regarding *Kodaira vanishing* in degree one.

## 1. CARTESIAN MORPHISMS AND EXTENSIONS OF GROUP ACTIONS

In this section, we recall Grothendieck’s notion of *cartesian morphisms* ([7], Exposé VI), and examine the problem of extending group actions along cartesian

morphisms, using the relation between second group cohomology and extensions of groups. Our motivation was to clarify and perhaps simplify some arguments of Rim [12], by putting them to this categorical setting.

Let  $p : \mathcal{F} \rightarrow \mathcal{E}$  be a functor between categories  $\mathcal{F}$  and  $\mathcal{E}$ . For each object  $S \in \mathcal{E}$ , we write  $\mathcal{F}(S) \subset \mathcal{F}$  for the subcategory of objects  $\xi$  with  $p(\xi) = S$ , and morphisms  $h : \xi \rightarrow \zeta$  with  $p(h) = \text{id}_S$ . The hom sets in this category are written as  $\text{Hom}_S(\xi, \zeta)$ . If  $\xi \in \mathcal{F}(S)$  and  $\xi' \in \mathcal{F}(S')$ , and  $S \rightarrow S'$  is a morphism in  $\mathcal{E}$ , we write  $\text{Hom}_{S \rightarrow S'}(\xi, \xi')$  for the set of morphisms  $f : \xi \rightarrow \xi'$  inducing the given  $S \rightarrow S'$ .

Let  $f : \xi \rightarrow \xi'$  be a morphism in  $\mathcal{F}$ , with induced morphism  $S \rightarrow S'$  in  $\mathcal{E}$ . One says that  $f : \xi \rightarrow \xi'$  is *cartesian* if the map

$$\text{Hom}_S(\zeta, \xi) \longrightarrow \text{Hom}_{S \rightarrow S'}(\zeta, \xi'), \quad h \longmapsto f \circ h$$

is bijective, for each  $\zeta \in \mathcal{F}(S)$ . Intuitively, this means that  $\xi$  is obtained from  $\xi'$  by “base-change” along  $S \rightarrow S'$ .

We also say that a cartesian morphism  $f : \xi \rightarrow \xi'$  is a *lifting* of  $\xi$  over  $S \rightarrow S'$ . Let  $\mathcal{Lif}(\xi, S')$  be the set of all such liftings; by abuse of notation, we suppress the morphism  $S \rightarrow S'$  from notation. The group elements  $\sigma \in \text{Aut}_S(\xi)$  act on  $\mathcal{Lif}(\xi, S')$  from the left by transport of structure, written as  ${}^\sigma f = f \circ \sigma^{-1}$ . We may regard  $\mathcal{Lif}(\xi, S')$  also as a category, where a morphism between  $f : \xi \rightarrow \xi'$  and  $g : \xi \rightarrow \xi'$  is an  $S'$ -morphism  $h : \xi' \rightarrow \xi'$  with  $h \circ f = g$ . Write  $\text{Lif}(\xi, S')$  for the set of isomorphism classes  $[f]$  of lifting. Obviously, the action of  $\text{Aut}_S(\xi)$  descends to an action  ${}^\sigma [f] = [f \circ \sigma^{-1}]$  from the left on  $\text{Lif}(\xi, S')$ .

Every  $S'$ -morphism  $\sigma' : \xi' \rightarrow \xi'$  yields the morphism  $\sigma' \circ f$  over  $S \rightarrow S'$ , which in turn corresponds to a unique  $S$ -morphism  $\sigma : \xi \rightarrow \xi$ , which makes the diagram

$$(3) \quad \begin{array}{ccc} \xi & \xrightarrow{f} & \xi' \\ \sigma \downarrow & & \downarrow \sigma' \\ \xi & \xrightarrow{f} & \xi' \end{array}$$

commutative. The map  $\sigma' \mapsto \sigma$  is compatible with compositions and respects identities, whence yields a homomorphism of groups

$$\text{Aut}_{S'}(\xi') \longrightarrow \text{Aut}_S(\xi), \quad \sigma' \longmapsto \sigma.$$

We call it the *restriction map*. Its kernel  $\text{Aut}_\xi(\xi')$  equals the group of automorphisms for the lifting  $f : \xi \rightarrow \xi'$ .

Now let  $G$  be a group acting on the object  $\xi \in \mathcal{F}(S)$ , via a homomorphism of groups  $G \rightarrow \text{Aut}_S(\xi)$ . We seek to extend this action on  $\xi \in \mathcal{F}(S)$  to an action on  $\xi' \in \mathcal{F}(S')$ . In other words, we want to complete the diagram

$$\begin{array}{ccc} & G & \\ & \downarrow & \\ \text{Aut}_{S'}(\xi') & \longrightarrow & \text{Aut}_S(\xi) \end{array}$$

with some dashed arrow. A necessary condition is that the image of  $G$  in  $\text{Aut}_S(\xi)$  is contained in the image of  $\text{Aut}_{S'}(\xi')$ . This can be reformulated as a fixed point problem:

**Proposition 1.1.** *The image of the homomorphism  $G \rightarrow \text{Aut}_S(\xi)$  is contained in the image of  $\text{Aut}_{S'}(\xi') \rightarrow \text{Aut}_S(\xi)$  if and only if  $[f] \in \text{Lif}(\xi, S')$  is a fixed point for the  $G$ -action.*

*Proof.* If the isomorphism class  $[f]$  is fixed, then for each  $\sigma \in G$ , there exists an isomorphism  $\sigma' : \xi' \rightarrow \xi'$  making the diagram (3) commutative. Since  $f$  is cartesian, the uniqueness of the arrow  $\sigma$  ensures that  $\sigma' \mapsto \sigma$  under the restriction map  $\text{Aut}_{S'}(\xi') \rightarrow \text{Aut}_S(\xi)$ . Conversely, if the image of  $G$  lies in the image of  $\text{Aut}_{S'}(\xi')$ , diagram (3) shows that the isomorphism class of the lifting  $f : \xi \rightarrow \xi'$  is  $G$ -fixed.  $\square$

Now suppose that  $[f] \in \text{Lif}(\xi, S')$  is a fixed point for the  $G$ -action, such that the image of  $G$  in  $\text{Aut}_S(\xi)$  lies in the image of  $\text{Aut}_{S'}(\xi')$ . Setting  $\tilde{G} = \text{Aut}_{S'}(\xi') \times_{\text{Aut}_S(\xi)} G$ , we get an induced extension of groups

$$(4) \quad 1 \longrightarrow \text{Aut}_\xi(\xi') \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

The splittings for this extension correspond to the extensions of the  $G$ -action to  $\xi'$ . To express this in cohomological terms, we now make the additional assumption that the kernel  $\text{Aut}_\xi(\xi')$  is abelian. This abelian group becomes a  $G$ -module, via  ${}^\sigma h = \Phi_\sigma \circ h \circ \Phi_\sigma^{-1}$ , where the  $\Phi_\sigma \in \tilde{G}$  map to  $\sigma \in G$ . This indeed satisfies the axioms for actions, and does not depend on the choices of  $\Phi_\sigma$ , because  $\text{Aut}_\xi(\xi')$  is abelian. Now the formula  $c_{\sigma, \tau} \Phi_{\sigma\tau} = \Phi_\sigma \Phi_\tau$  defines a cochain  $c : G^2 \rightarrow \text{Aut}_\xi(\xi')$ . As explained in [1], Chapter IV, Section 3, this cochain is a cocycle, and the resulting cohomology class

$$[\tilde{G}] \in H^2(G, \text{Aut}_\xi(\xi'))$$

does not depend on the choice of the  $\Phi_\sigma$ . Moreover, the extension of groups (4) splits if and only if  $[\tilde{G}] = 0$ . In this case, the extension is a semidirect product  $\text{Aut}_\xi(\xi') \rtimes G$ . Indeed, the group  $H^2(G, \text{Aut}_\xi(\xi'))$  corresponds to isomorphism classes of group extensions of  $G$  by  $\text{Aut}_\xi(\xi')$  inducing the given  $G$ -module structure. Summing up, we have shown the following “abstract nonsense” result:

**Theorem 1.2.** *Let  $p : \mathcal{F} \rightarrow \mathcal{E}$  be a functor,  $f : \xi \rightarrow \xi'$  be a cartesian morphism in  $\mathcal{F}$ , and  $S \rightarrow S'$  be the resulting morphism in  $\mathcal{E}$ . Let  $G \rightarrow \text{Aut}_S(\xi)$  be a homomorphism of groups, and assume that the group  $\text{Aut}_\xi(\xi')$  is abelian. Then the  $G$ -action on  $\xi \in \mathcal{F}(S)$  extends to a  $G$ -action on  $\xi' \in \mathcal{F}(S')$  if and only if the following two conditions holds:*

- (i) *The isomorphism class  $[f] \in \text{Lif}(\xi, S')$  is fixed under the  $G$ -action.*
- (ii) *The resulting cohomology class  $[\tilde{G}] \in H^2(G, \text{Aut}_\xi(\xi'))$  is trivial.*

Of particular practical importance are the *fibered categories*  $p : \mathcal{F} \rightarrow \mathcal{E}$ . This means that for each morphism  $S \rightarrow S'$  in  $\mathcal{E}$  and each object  $\xi' \in \mathcal{F}(S')$ , there is a cartesian morphism  $f : \xi \rightarrow \xi'$  in  $\mathcal{F}_{S \rightarrow S'}$ , and the composition of cartesian morphisms in  $\mathcal{F}$  is again cartesian. A *cleavage* is the choice, for each  $S \rightarrow S'$  and  $\xi' \in \mathcal{F}(S')$ , of such a cartesian morphism  $f : \xi \rightarrow \xi'$ , which are called *transport morphisms*. If the transport morphisms for identities are identities, one calls the cleavage *normalized*. We also write  $\xi'|_S = \xi$  for the domains. Intuitively, one should regard it as a “restriction”, “pull-back” or “base-change” of  $\xi'$  along  $S \rightarrow S'$ . In fact, the transport morphisms induce restriction or pull-back functors

$$\mathcal{F}(S') \longrightarrow \mathcal{F}(S), \quad \xi' \longmapsto \xi'|_S.$$

In particular, for every amalgamated sum  $S' \amalg_S S''$  in  $\mathcal{E}$ , we get a functor

$$(5) \quad \mathcal{F}(S' \amalg_S S'') \longrightarrow \mathcal{F}(S') \times_{\mathcal{F}(S)} \mathcal{F}(S''), \quad \xi \longmapsto (\xi|_{S'}, \xi|_{S''}, \varphi),$$

where  $\varphi : (\xi|_{S'})|_S \longrightarrow (\xi|_{S''})|_S$  is the unique *comparison isomorphism*, compare [7], Exposé VI, Proposition 7.2, and the right hand side in (5) is the *2-fiber product of categories*, as explained in [18], Appendix C.

A *category fibered in groupoids* is a fibered category  $p : \mathcal{F} \rightarrow \mathcal{E}$  so that the categories  $\mathcal{F}(S)$ , with  $S \in \mathcal{E}$  are groupoids. These are the fibered categories that occur in moduli problems or deformation theory. They have the property that every morphism in  $\mathcal{F}$  is cartesian, compare [7], Exposé VI, Remark after Definition 6.1.

## 2. TORSORS WITH A GROUP OF OPERATORS

In this section we set up further notation, recall Serre's interpretation of first group cohomology in terms of torsors [15], §5.2, and relate it to fixed point problems. Let  $G$  be a group that acts from the left via automorphisms on another group  $T$  and a set  $L$ . We write these actions as  $t \mapsto {}^\sigma t$  and  $\xi \mapsto {}^\sigma \xi$ , where  $\sigma \in G$ . Suppose we have an action on the right

$$\mu : L \times T \longrightarrow L, \quad (\xi, t) \longmapsto \xi \cdot t,$$

such that the set  $L$  is a *principal homogeneous space* for the group  $T$ , that is, a right *T-torsor*. In other words, the set  $L$  is non-empty, and for each point  $\xi_0 \in L$  the resulting map  $T \rightarrow L$ ,  $t \mapsto \xi_0 \cdot t$  is bijective. We assume throughout that this action is compatible with the  $G$ -action in the sense

$$\sigma(\xi \cdot t) = {}^\sigma \xi \cdot {}^\sigma t,$$

for all  $\sigma \in G$ ,  $\xi \in L$  and  $t \in T$ . One says that the  $T$ -torsor  $L$  is *endowed with a group of operators*  $G$ . They are the objects of a category, where the morphisms  $(L, T) \rightarrow (L', T')$  are pairs  $(f, h)$ , where  $f : L \rightarrow L'$  is a  $G$ -equivariant maps, and  $h : T \rightarrow T'$  is a  $G$ -equivariant homomorphism, which satisfy

$$f(\xi \cdot t) = f(\xi) \cdot h(t).$$

In this situation, we want to decide whether or not the  $G$ -set  $L$  has a fixed point. To this end, one may construct a cohomology class  $[L] \in H^1(G, T)$  as follows: Choose some  $\xi \in L$ . Then the equation  ${}^\sigma \xi = \xi \cdot t_\sigma$  defines a map

$$G \longrightarrow T, \quad \sigma \longmapsto t_\sigma,$$

which we regard as a 1-cochain. The equation

$$\xi \cdot t_{\eta\sigma} = {}^{\eta\sigma} \xi = {}^\eta(\xi \cdot t_\sigma) = {}^\eta \xi \cdot {}^\eta t_\sigma = (\xi \cdot t_\eta) \cdot {}^\eta t_\sigma = \xi \cdot (t_\eta {}^\eta t_\sigma)$$

implies  $t_{\eta\sigma} = t_\eta {}^\eta t_\sigma$ , and it follows that the cochain is a cocycle. For every other point  $\xi' \in L$ , the equation  ${}^\sigma \xi' = \xi' \cdot t'_\sigma$  defines another cocycle  $\sigma \mapsto t'_\sigma$ . We have  $\xi' \cdot s = \xi$  for some  $s \in T$ , and thus

$$\xi' \cdot (t'_\sigma {}^\sigma s) = {}^\sigma \xi' \cdot {}^\sigma s = {}^\sigma(\xi' \cdot s) = {}^\sigma \xi = \xi \cdot t_\sigma = (\xi' \cdot s) \cdot t_\sigma = \xi' \cdot (st_\sigma).$$

It follows that  $t'_\sigma = st_\sigma {}^\sigma(s^{-1})$ , whence the two cocycles are cohomologous. We thus get a well-defined cohomology class

$$[L] \in H^1(G, T).$$

In this general non-abelian setting, we regard  $H^1(G, T)$  as a pointed set, where the distinguished point  $\star \in H^1(G, T)$  is the cohomology class of the constant cocycle  $\sigma \mapsto e$ . It is also called the *trivial cohomology class*. According to [15], Proposition 33, this gives a pointed bijection between the set of isomorphism classes of  $T$ -torsors  $L$  with a group of operators  $G$ , and the set  $H^1(G, T)$ . We need the following consequence:

**Lemma 2.1.** *The cohomology class  $[L] \in H^1(G, T)$  is trivial if and only if the set of fixed points  $L^G$  is nonempty.*

*Proof.* The condition is clearly sufficient: If  $\xi \in L$  is  $G$ -fixed, then the resulting cocycle is  $t_\sigma = e$ , so the cohomology class  $[L]$  is trivial. Conversely suppose that the cocycle  $t_\sigma$  attached to a point  $\xi \in L$  satisfies  $st_\sigma^\sigma(s^{-1}) = e$  for some  $s \in T$ . Then

$$\sigma(\xi \cdot s^{-1}) = {}^\sigma \xi \cdot {}^\sigma s^{-1} = \xi \cdot t_\sigma \cdot {}^\sigma s^{-1} = (\xi \cdot s^{-1}) \cdot (st_\sigma^\sigma s^{-1}) = \xi \cdot s^{-1},$$

whence  $\xi' = \xi \cdot s^{-1}$  is the desired fixed point.  $\square$

### 3. DEFORMATION CATEGORIES AND GROUP ACTIONS

Let  $k$  be a field of characteristic  $p \geq 0$ , and let  $\Lambda$  be a complete local noetherian ring with residue field  $k = \Lambda/\mathfrak{m}_\Lambda$ . We write  $(\text{Art}_\Lambda)$  for the category of local Artin  $\Lambda$ -algebras  $A$  such that the induced map  $k = \Lambda/\mathfrak{m}_\Lambda \rightarrow A/\mathfrak{m}_A$  on residue fields is bijective. Let  $\mathcal{F} \rightarrow (\text{Art}_\Lambda)^{\text{op}}$  is a category fibered in groupoids satisfying the *Rim-Schlessinger condition*. Recall that the latter means that for every cartesian square

$$\begin{array}{ccc} A' \times_A A'' & \longrightarrow & A'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

in the category  $(\text{Art}_\Lambda)$ , the resulting functor

$$\mathcal{F}(A' \times_A A'') \longrightarrow \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$$

is an equivalence of categories. Note that this functor corresponds to (5), and is actually defined with the help of a chosen cleavage, but the fact that it is an equivalence does not depend on this choice. Such a condition was first introduced by Schlessinger [13], who considered functors of Artin rings, and extended to fibered categories by Rim [11]. Following Talpo and Vistoli [18], we say that such a category fibered in groupoids  $\mathcal{F} \rightarrow (\text{Art}_\Lambda)^{\text{op}}$  is a *deformation category*.

Note that one should regard the opposite category  $(\text{Art}_\Lambda)^{\text{op}}$  as a full subcategory of the category  $(\text{Sch}/\Lambda)$  of schemes. The morphisms in this category are thus  $\text{Spec}(A) \rightarrow \text{Spec}(A')$ , and correspond to algebra homomorphisms  $A' \rightarrow A$ . The transport morphisms over an algebra homomorphism  $B \rightarrow C$ , that is  $\text{Spec}(C) \rightarrow \text{Spec}(B)$ , could also be written in tensor product notation  $\zeta \otimes_B C \rightarrow \zeta$  instead of  $\zeta|_C \rightarrow \zeta$ . Indeed, in praxis the deformation category  $\mathcal{F} \rightarrow (\text{Art}_\Lambda)^{\text{op}}$  often consists of flat morphisms  $X \rightarrow \text{Spec}(C)$  of certain schemes, and the transport morphisms are given by projections  $\text{pr}_1 : X \otimes_B C = X \times_{\text{Spec}(B)} \text{Spec}(C) \rightarrow X$ .

Let  $A \in (\text{Art}_\Lambda)$ , and  $\xi \in \mathcal{F}(A)$  be some object. Suppose that  $G$  is a group endowed with a homomorphism  $G \rightarrow \text{Aut}_A(\xi)$ . In other words,  $G$  acts on the object

$\xi \in \mathcal{F}$  so that the induced action on  $A \in (\text{Art}_\Lambda)$  is trivial. In what follows,

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0$$

is a *small extension* with ideal  $I \subset A'$ . This means that  $I \cdot \mathfrak{m}_\Lambda = 0$ , so we may regard the  $\Lambda$ -module  $I$  simply as a  $k$ -vector space.

We now ask whether there exist a lifting  $f : \xi \rightarrow \xi'$  over  $\text{Spec}(A) \subset \text{Spec}(A')$  to which the  $G$ -action extends. Of course, the category of all liftings may be empty, and then nothing useful can be said. But if one assumes that some lift exists, a natural question is whether some possibly different lifting can be endowed with a  $G$ -action. To this end, we apply Theorem 1.2 to our situation. Recall that  $\mathcal{Lif}(\xi, A')$  denotes the category of all liftings  $f : \xi \rightarrow \xi'$  over  $\text{Spec}(A) \subset \text{Spec}(A')$ , and let  $\text{Lif}(\xi, A')$  be the set of isomorphism classes  $[f]$ , endowed with the canonical  $G$ -action

$$\sigma[f] = [f \circ \sigma^{-1}].$$

To proceed, choose a morphism  $\xi_0 \rightarrow \xi$  over  $\text{Spec}(k) \subset \text{Spec}(A)$ , and consider the resulting *tangent space*

$$T_{\xi_0} \mathcal{F} = \text{Lif}(\xi_0, k[\epsilon]),$$

where  $\epsilon$  denotes an indeterminate subject to the relation  $\epsilon^2 = 0$ . In other words,  $k[\epsilon] \in (\text{Art}_\Lambda)$  is the *ring of dual numbers*, with ideal  $k\epsilon$ .

The Rim–Schlessinger condition ensures that the functor  $I \mapsto \text{Lif}(\xi_0, k[I])$  of  $k$ -vector spaces  $I$  preserves finite products, and as a consequence  $\text{Lif}(\xi_0, k[I])$  and in particular the tangent spaces  $T_{\xi_0} \mathcal{F}$  acquire the structure of an abelian group, and actually become  $k$ -vector spaces. As explained in [18], Appendix A, the natural transformation in  $I$  given by

$$(6) \quad I \otimes_k \text{Lif}(\xi_0, k[\epsilon]) \longrightarrow \text{Lif}(\xi_0, k[I]), \quad v \otimes [\xi_0 \xrightarrow{f} \psi] \longmapsto [\xi_0 \xrightarrow{\alpha} \psi|_{k[I]}]$$

is a natural isomorphism. Here the object  $\psi|_{k[I]}$  arises from the transport morphism  $\psi|_{k[I]} \rightarrow \psi$  over the morphism  $\text{Spec}(k[I]) \rightarrow \text{Spec}(k[\epsilon])$  induced from the linear map  $k\epsilon \rightarrow I$  with  $\epsilon \mapsto v$ , and  $\alpha : \xi_0 \rightarrow \psi|_{k[I]}$  is the transport morphism over the inclusion  $\text{Spec}(k) \subset \text{Spec}(k[I])$  given by  $I \rightarrow 0$ . Clearly, this natural isomorphism respects the action of the  $\text{Aut}_k(\xi_0)$ , where the group elements  $\sigma \in \text{Aut}_k(\xi_0)$  act via transport of structure

$$v \otimes [\xi_0 \xrightarrow{f} \psi] \longmapsto v \otimes [\xi_0 \xrightarrow{f\sigma^{-1}} \psi] \quad \text{and} \quad [\xi_0 \xrightarrow{\alpha\sigma^{-1}} \psi|_{k[I]}].$$

Note that the action on  $v \in I$  is trivial. In what follows, we regard the above natural isomorphism as an identification  $I \otimes_k T_{\xi_0}(\mathcal{F}) = \text{Lif}(\xi_0, k[I])$ . Furthermore, the underlying abelian group acts on  $\text{Lif}(\xi, A')$  in a canonical way, via some

$$(7) \quad \text{Lif}(\xi, A') \times (I \otimes_k T_{\xi_0} \mathcal{F}) \longrightarrow \text{Lif}(\xi, A')$$

recalled in (8) below. The  $G$ -action on  $\xi$  induces a  $G$ -action on  $\xi_0$ , and we also get a linear  $G$ -action on the tangent space  $T_{\xi_0} \mathcal{F}$ , as described above.

**Proposition 3.1.** *Suppose the set  $L = \text{Lif}(\xi, A')$  is non-empty. With respect to the action of  $T = I \otimes_k T_{\xi_0} \mathcal{F}$ , the set  $L$  is a  $T$ -torsor with a group of operators  $G$ .*

*Proof.* As explained in [18], Theorem 3.15, the Rim–Schlessinger condition ensures that the set  $L$  becomes a  $T$ -torsor. Our task is merely to check that this structure is compatible with the  $G$ -actions. To this end, we have to unravel the action of  $T$

on  $L$ . Let  $f : \xi \rightarrow \xi'$  be lifting of  $\xi \in \mathcal{F}(A)$  over  $A'$ , and  $g : \xi_0 \rightarrow \tilde{\xi}$  be a lifting of  $\xi_0 \in \mathcal{F}(k)$  over the ring of dual numbers  $\hat{A} = k[I]$  with ideal  $I$ . We have to describe  $[f] + [g] \in \text{Lif}(\xi, A')$  and understand how the group  $G$  acts on this.

To proceed, choose a cleavage for the fibered category  $\mathcal{F} \rightarrow (\text{Art}_\Lambda)^{\text{op}}$ . In other words, we fix for each object  $\zeta \in \mathcal{F}(C)$  and each homomorphism  $B \rightarrow C$  a transport morphism  $\zeta|_C \rightarrow \zeta$  over  $\text{Spec}(C) \rightarrow \text{Spec}(B)$  and regard the domain  $\zeta|_C$  as the restriction of  $\zeta$ . We do this so that  $\xi_0 = \xi|_k$  holds. In what follows, we simply write  $\alpha : \zeta|_C \rightarrow \zeta$  for these transport morphisms. Now the morphism  $f$  and  $g$  correspond to isomorphisms

$$\bar{f} : \xi \longrightarrow \xi'|_A \quad \text{and} \quad \bar{g} : \xi_0 \longrightarrow \tilde{\xi}|_k,$$

and we can form the composite morphism

$$\psi : \xi'|_k \xrightarrow{\bar{f}^{-1}|_k} \xi_0 \xrightarrow{\bar{g}} \tilde{\xi}|_k.$$

This gives us a triple  $(\xi', \tilde{\xi}, \psi)$ , which we regard as an object in the fiber product category

$$\mathcal{F}(A') \times_{\mathcal{F}(k)} \mathcal{F}(k[I]).$$

Now recall that we have isomorphisms of rings

$$A' \times_A A' \longrightarrow A' \times_k (k[I]), \quad (a_1, a_2) \longmapsto (a_1, (a_1 \bmod \mathfrak{m}_{A'}, a_2 - a_1)).$$

Here we use  $k[I] = k \oplus I$ , and write  $a_1 \bmod \mathfrak{m}_{A'}$  for the residue class in  $k$ , and regard  $a_2 - a_1$  as element from  $I$ . The Rim-Schlessinger condition yields equivalences of categories

$$\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A') \longleftarrow \mathcal{F}(A' \times_A A') \longrightarrow \mathcal{F}(A' \times_k k[I]) \longrightarrow \mathcal{F}(A') \times_{\mathcal{F}(k)} \mathcal{F}(k[I]),$$

where the restriction functors are defined in terms of the chosen cleavage. Choose adjoint equivalences, to get an equivalence of categories

$$\mathcal{F}(A') \times_{\mathcal{F}(k)} \mathcal{F}(k[I]) \longrightarrow \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A').$$

We may choose this functor so that it commutes with the projections onto the first factor  $\mathcal{F}(A')$ . Applying this functor to the object  $(\xi', \tilde{\xi}, \psi)$  yields an object  $(\xi', \zeta', \varphi)$ , where  $\xi', \zeta' \in \mathcal{F}(A')$  and  $\varphi : \xi'|_A \rightarrow \zeta'|_A$  is an isomorphism. In turn, we get a lifting from the composite morphism

$$(8) \quad h : \xi \xrightarrow{\bar{f}} \xi'|_A \xrightarrow{\varphi} \zeta'|_A \xrightarrow{\alpha} \zeta'.$$

Here  $\alpha : \zeta'|_A \rightarrow \zeta'$  is a transport morphism. The  $T$ -action on  $L$  is given by  $[f] + [g] = [h]$ , as explained in [18], Theorem 3.15.

Now we are in the position to unravel the  $G$ -action. Let  $\sigma \in G$ . By definition,  $\sigma[f] = [f \circ \sigma^{-1}]$  and  $\sigma[g] = [g \circ \sigma^{-1}]$ . Using  $f \circ \sigma^{-1}$  and  $g \circ \sigma^{-1}$  rather than  $f$  and  $g$  in the preceding paragraph, we get

$$\overline{f \circ \sigma^{-1}} = \bar{f} \circ \sigma^{-1}, \quad \text{and} \quad \overline{g \circ \sigma^{-1}} = \bar{g} \circ (\sigma^{-1}|_k) = \bar{g} \circ (\sigma|_k)^{-1},$$

which implies

$$\overline{g \circ \sigma^{-1}} \circ \overline{f \circ \sigma^{-1}}^{-1}|_k = \bar{g} \circ (\sigma|_k)^{-1} \circ (\sigma|_k) \circ \bar{f}^{-1}|_k = \bar{g} \circ \bar{f}^{-1}|_k.$$



It follows that the resulting morphism  $\psi : \xi'|_k \rightarrow \tilde{\xi}$  is the same, whether computed with  $f \circ \sigma^{-1}$  and  $g \circ \sigma^{-1}$ , or with  $f$  and  $g$ . In turn, the image of the object  $(\xi', \tilde{\xi}, \psi)$  remains the object  $(\xi', \zeta', \varphi)$ . The resulting lifting is thus given by the composite

$$\xi \xrightarrow{\bar{f}\sigma^{-1}} \xi'|_A \xrightarrow{\varphi} \zeta'|_A \xrightarrow{\alpha} \zeta',$$

which equals  $h \circ \sigma^{-1}$ . This shows that  $\sigma[f] + \sigma[g] = \sigma[h]$ . In other words, the  $T$ -torsor  $L$  is endowed with a group of operators  $G$ .  $\square$

As described in Section 2, this  $L$ -torsor  $T$  endowed with a group of operators  $G$  yields a cohomology class

$$[\text{Lif}(\xi, A')] \in H^1(G, I \otimes_k T_{\xi_0} \mathcal{F}),$$

and Lemma 2.1 immediately gives:

**Theorem 3.2.** *Suppose  $\text{Lif}(\xi, A')$  is non-empty. Then there is a  $G$ -fixed isomorphism class  $[f] \in \text{Lif}(\xi, A')$  of liftings  $f : \xi \rightarrow \xi'$  over  $\text{Spec}(A) \subset \text{Spec}(A')$  if and only if the cohomology class  $[\text{Lif}(\xi, A')] \in H^1(G, I \otimes_k T_{\xi_0} \mathcal{F})$  is trivial.*

Now suppose that there exists a lifting  $f : \xi \rightarrow \xi'$  whose isomorphism class  $[f] \in \text{Lif}(\xi, A')$  is fixed under the  $G$ -action. As discussed in Section 1, we get an extension of groups

$$(9) \quad 1 \longrightarrow \text{Aut}_{\xi}(\xi') \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,$$

and this extension of groups splits if and only if the  $G$ -action on  $\xi$  extends to  $\xi'$ .

Now choose a morphism  $\xi_{k[\epsilon]} \rightarrow \xi_0$  over the morphism  $\text{Spec}(k[\epsilon]) \rightarrow \text{Spec}(k)$  corresponding to the canonical inclusion  $k \subset k[\epsilon]$ . As explained in [18], Proposition 4.5, we have a canonical identification

$$(10) \quad I \otimes_k \text{Aut}_{\xi_0}(\xi_{k[\epsilon]}) = \text{Aut}_{\xi}(\xi'),$$

and this group carries the structure of  $k$ -vector space. In particular, it is abelian. In fact, (10) is an incarnation of (7), for the deformation theory  $\mathcal{A} \rightarrow (\text{Art}_{\Lambda})^{\text{op}}$  whose objects over  $A$  are the automorphisms of  $\xi_0|_A$ , as explained in [18], Section 4.

Since the isomorphism class of  $f : \xi \rightarrow \xi'$  is  $G$ -fixed, we have a natural  $G$ -action on  $\text{Aut}_{\xi'}(\xi)$ , coming from the extension (9) or equivalently from diagram (3). The same applies for  $\xi_0 \rightarrow \xi_{k[\epsilon]}$ , and we thus get a  $G$ -action on  $\text{Aut}_{\xi_0}(\xi_{k[\epsilon]})$ . Taking the trivial  $G$ -action on  $I$ , both sides in (10) acquire a  $G$ -action, and these actions coincide under the identification. We thus may regard the extension class for (9) as an element in

$$[\tilde{G}] \in H^2(G, \text{Aut}_{\xi}(\xi')) = H^2(G, I \otimes_k \text{Aut}_{\xi_0}(\xi_{k[\epsilon]})).$$

Now Theorem 1.2 yields:

**Theorem 3.3.** *Suppose that  $\text{Lif}(\xi, A')^G$  is non-empty, and let  $f : \xi \rightarrow \xi'$  be a lifting over  $\text{Spec}(A) \subset \text{Spec}(A')$  whose isomorphism class is fixed under the  $G$ -action. Then the  $G$ -action on  $\xi$  extends to an action on  $\xi'$  if and only if the resulting cohomology class  $[\tilde{G}] \in H^2(G, I \otimes_k \text{Aut}_{\xi_0}(\xi_{k[\epsilon]}))$  vanishes.*

In the following applications, we assume that the group  $G$  is finite, and write  $n = \text{ord}(G)$  for its order.

**Proposition 3.4.** *Suppose  $\text{Lif}(\xi, A')$  is non-empty and that the group order  $n \geq 1$  is invertible in the residue field  $k$ . Then the  $G$ -action on  $\xi$  extends to an action on  $\xi'$  for some lifting  $f : \xi \rightarrow \xi'$ .*

*Proof.* The cohomology group  $H^1(G, I \otimes_k T_{\xi_0} \mathcal{F})$  is a vector space over the field  $k$ , and at the same time an abelian group annihilated by  $n = \text{ord}(G)$ . Thus it must be the zero group, and Theorem 3.2 ensures that there is a lifting  $\xi \rightarrow \xi'$  over  $\text{Spec}(A) \subset \text{Spec}(A')$  whose isomorphism class is fixed under the  $G$ -action. Arguing as above, the cohomology group  $H^2(G, I \otimes_k \text{Aut}_{\xi_0}(\xi_{k[\epsilon]}))$  vanishes, and Theorem 3.3 tells us that we may extend the  $G$ -action from  $\xi$  to  $\xi'$ .  $\square$

**Proposition 3.5.** *Suppose  $\text{Lif}(\xi, A')$  is non-empty and that the residue field  $k$  has characteristic  $p > 0$ . Let  $P \subset G$  be a Sylow  $p$ -subgroup. Then  $\text{Lif}(\xi, A')$  has a  $G$ -fixed point if and only if it has a  $P$ -fixed point. Moreover, for each  $[\xi'] \in \text{Lif}(\xi, A')^G$ , the  $G$ -action on  $\xi$  extends to  $\xi'$  if and only if the  $P$ -action extends.*

*Proof.* According to [1], Chapter III, Proposition 10.4 the restriction map

$$H^1(G, I \otimes_k T_{\xi_0} \mathcal{F}) \longrightarrow H^1(P, I \otimes_k T_{\xi_0} \mathcal{F})$$

is injective, and the first assertion follows from Theorem 3.2. If there is a lifting  $\xi \rightarrow \xi'$  whose isomorphism class is  $G$ -invariant, we again have an injective restriction map

$$H^2(G, I \otimes_k \text{Aut}_{\xi_0}(\xi_{k[\epsilon]})) \longrightarrow H^2(P, I \otimes_k \text{Aut}_{\xi_0}(\xi_{k[\epsilon]})),$$

and the second assertion follows from Theorem 3.3.  $\square$

Recall that a finitely generated free  $kP$ -modules  $V$  have trivial cohomology groups  $H^i(P, V)$ , for all  $i \geq 1$ . We thus get:

**Corollary 3.6.** *Assumptions as in the proposition. Then  $\text{Lif}(\xi, A')$  has a  $G$ -fixed point if  $\text{Lif}(\xi, A')$  is free as  $kP$ -module. Moreover, for each  $[\xi'] \in \text{Lif}(\xi, A')^G$ , the  $G$ -action on  $\xi$  extends to  $\xi'$  if  $\text{Aut}_{\xi_0}(\xi_{k[\epsilon]})$  is free as  $kP$ -module.*

In some sense, this seems to be the best possible general result: According to [1], Chapter VI, Theorem 8.5, for every finite  $p$ -group  $P$  and every field  $k$  of characteristic  $p > 0$ , the following holds for  $kP$ -modules  $V$ :

$$H^1(P, V) = 0 \iff H^2(P, V) = 0 \iff \text{the } kP\text{-module } V \text{ is free.}$$

If  $P$  is cyclic of order  $p^\nu$  and  $V$  is finitely generated, then the action of any generator  $\sigma \in P$  can be viewed as a direct sum  $\sigma = J_{r_1} \oplus \dots \oplus J_{r_m}$  of Jordan matrices  $J_r \in \text{GL}_r(k)$  with eigenvalue  $\lambda = 1$ . In this case, the  $kP$ -module  $V$  is free if and only if all summands have maximal size  $r_i = p^\nu$ .

#### 4. LIFTINGS AND ALGEBRAIZATION OF FINITE ÉTALE COVERINGS

Let  $\Lambda$  be an adic noetherian ring, with ideal of definition  $\mathfrak{a} \subset \Lambda$ . In other words, the ring  $\Lambda$  is noetherian, and separated and complete with respect to the  $\mathfrak{a}$ -adic topology. For example, the ring  $\Lambda$  could be a complete local noetherian ring. Let  $Y \rightarrow \text{Spec}(\Lambda)$  be a proper morphism, and set  $Y_0 = Y \otimes_\Lambda \Lambda/\mathfrak{a}$ . Write  $(\text{Sch}/Y)$  for the category of  $Y$ -schemes, and let  $(\text{FÉt}/Y)$  the full subcategory whose objects are the  $Y$ -schemes whose structure morphism  $\pi : X \rightarrow Y$  is finite and étale. The goal of this section is to establish the following:

**Theorem 4.1.** *In the above situation, the restriction functor*

$$(\mathrm{FEt}/Y) \longrightarrow (\mathrm{FEt}/Y_0), \quad X \longmapsto X \times_Y Y_0$$

*is an equivalence of categories.*

*Proof.* The main task is to show that the restriction functor is essentially surjective. To do so, let  $\pi_0 : X_0 \rightarrow Y_0$  be a finite étale morphism. Set  $\Lambda_n = \Lambda/\mathfrak{a}^{n+1}$ , and consider the infinitesimal neighborhoods  $Y_n = Y \otimes_{\Lambda} \Lambda_n$ . According to [6], Theorem 18.1.2, the restriction functors  $(\mathrm{FEt}/Y_m) \rightarrow (\mathrm{FEt}/Y_n)$  are equivalences of categories for all  $m \geq n$ . Inductively, we choose a finite étale  $\pi_n : X_n \rightarrow Y_n$  and cartesian diagrams

$$(11) \quad \begin{array}{ccc} X_n & \longrightarrow & X_{n+1} \\ \pi_n \downarrow & & \downarrow \pi_{n+1} \\ Y_n & \longrightarrow & Y_{n+1}. \end{array}$$

This gives a direct system  $(X_n)_{n \geq 0}$  of  $Y$ -schemes, and in turn a locally ringed space  $\mathfrak{X} = (X_0, \mathcal{O}_{\mathfrak{X}})$  whose structure sheaf is  $\mathcal{O}_{\mathfrak{X}} = \varprojlim \mathcal{O}_{X_n}$ . Then  $\mathfrak{X}$  is a formal scheme, according to [3], Proposition 10.6.3. Let  $\mathcal{S}_{ji}$  be the kernels of the canonical surjections  $u_{ji} : \mathcal{O}_{X_i} \rightarrow \mathcal{O}_{X_j}$ . The closed subscheme  $Y_j \subset Y_i$  corresponds to the coherent ideal  $\mathfrak{a}^{i+1} \mathcal{O}_{Y_i}$ . Since the diagrams (11) are cartesian, we have  $\mathcal{S}_{ji} = \mathfrak{a}^{j+1} \mathcal{O}_{Y_i}$ . Setting  $\mathcal{S}_i = \mathcal{S}_{0i}$ , we get  $\mathcal{S}_{ji} = \mathcal{S}_i^{j+1}$ . Thus [3], Corollary 10.6.4 applies, and we infer that the formal scheme  $\mathfrak{X}$  is adic and noetherian.

Likewise, we define  $\mathfrak{Y} = (Y_0, \mathcal{O}_{\mathfrak{Y}})$  via  $\mathcal{O}_{\mathfrak{Y}} = \varprojlim \mathcal{O}_{Y_n}$ , which is also an adic noetherian scheme, in fact the formal completion of  $Y$  along  $Y_0 \subset Y$ . The diagrams (11) yield a morphism of locally ringed spaces  $\pi_{\infty} : \mathfrak{X} \rightarrow \mathfrak{Y}$ , by [3], Corollary 10.6.11. According to [5], Proposition 4.8.1 this morphism  $\pi_{\infty} : \mathfrak{X} \rightarrow \mathfrak{Y}$  is finite. By assumption, the formal scheme  $\mathfrak{Y}$  is algebraizable. According to [5], Proposition 5.4.4 the same holds for  $\mathfrak{X}$ . In other words, there is a proper morphism  $X \rightarrow \mathrm{Spec}(\Lambda)$  of schemes so that our formal scheme  $\mathfrak{X}$  is isomorphic to the formal completion along  $X_0 = X \otimes_{\Lambda} \Lambda_0$ .

The morphism  $\pi_{\infty} : \mathfrak{X} \rightarrow \mathfrak{Y}$  comes from a unique  $\Lambda$ -morphism  $\pi : X \rightarrow Y$ , according to [5], Theorem 5.4.1. In fact, the algebraization is the relative spectrum  $X = \mathrm{Spec}(\mathcal{A})$  of some finite  $\mathcal{O}_Y$ -algebra  $\mathcal{A}$  whose formal completion becomes the finite  $\mathcal{O}_{\mathfrak{Y}}$ -algebra  $\mathcal{B} = (\pi_{\infty})_*(\mathcal{O}_{\mathfrak{X}}) = \varprojlim \mathcal{O}_{X_n}$ . This ensures that the morphism  $\pi : X \rightarrow Y$  is finite. Moreover, the base-change  $X \otimes_{\Lambda} \Lambda_n$  is isomorphic to  $X_n$ . In particular, we recover the original finite étale covering  $\pi_0 : X_0 \rightarrow Y_0$ .

We still have to check that the morphism  $\pi : X \rightarrow Y$  is étale. By the Local Flatness Criterion ([8], Section 20, Theorem 49), the finite morphism of locally ringed spaces  $\pi_{\infty} : \mathfrak{X} \rightarrow \mathfrak{Y}$  is flat. It follows that the  $\mathcal{O}_{\mathfrak{Y}}$ -module  $\mathcal{B}$  is locally free of finite rank. In light of [3], Corollary 10.8.15 the same holds for the coherent  $\mathcal{O}_Y$ -module  $\mathcal{A}$ . Thus  $\pi : X \rightarrow Y$  is flat. Consider the map

$$(12) \quad \mathcal{A} \longrightarrow \mathcal{A}^{\vee}, \quad a \longmapsto (a' \longmapsto \mathrm{Tr}_{\mathcal{A}/\mathcal{O}_Y}(x \mapsto aa'x))$$

associated to the trace map. The finite flat  $\mathcal{O}_Y$ -algebra  $\mathcal{A}$  is étale if and only if the trace map is surjective ([6], Proposition 18.2.3). Clearly, the induced map  $\mathcal{B} \rightarrow \mathcal{B}^{\vee}$

on formal completion is associated to the trace map for  $\mathcal{B} = \varprojlim \mathcal{O}_{X_n}$ , and the latter is surjective. According to [5], Corollary 5.1.3 the map (12) is already surjective.

Summing up, we have shown that  $\pi : X \rightarrow Y$  is finite and étale, and it induces the given  $\pi_0 : X_0 \rightarrow Y_0$ . It remains to show that the restriction functor  $X \mapsto X_0$  is fully faithful. But this follows easily from [7], Exposé 1, Corollary 8.4 together with [5], Theorem 5.4.1.  $\square$

Note that the above result strengthens [7], Exposé 1, Corollary 8.4, because it says that the finite étale covering  $\pi_0 : X_0 \rightarrow Y_0$  not only admits a formal lifting, but even an algebraic lifting.

Now suppose that  $\Lambda$  is a complete local noetherian ring with residue field  $k = \Lambda/\mathfrak{m}_\Lambda$ , and let  $X_0$  be a proper  $k$ -scheme endowed with a free action of a finite group  $G$ . We see that if the quotient  $Y_0 = X_0/G$  admits a lifting  $Y \rightarrow \text{Spec}(\Lambda)$ , then also  $X_0$  admits a lifting  $X \rightarrow \text{Spec}(\Lambda)$ , and the  $G$ -actions extend to all infinitesimal neighborhoods  $X_n$  and thus to  $X$ . Consequently, the first obstruction in Theorem 3.2 vanishes, and one always may choose extensions so that the second obstruction in Theorem 3.3 vanishes as well.

## 5. PULLBACKS OF PRE-TANGO STRUCTURES

In this section, we observe that pre-Tango structures are preserved under finite generically étale morphisms, and state some consequences concerning Kodaira vanishing.

Let  $k$  be an algebraically closed ground field of characteristic  $p > 0$ . Suppose  $Y$  is an integral smooth projective scheme, with generic point  $\eta \in Y$  and function field  $F = k(Y) = \kappa(\eta)$ . Let  $\iota : \text{Spec}(F) \rightarrow Y$  be the inclusion of the generic point. For arbitrary divisors  $D \in \text{Div}(Y)$ , one gets an injection of quasicoherent sheaves  $\Omega_{Y/k}^1(-D) \subset \iota_* \iota^* \Omega_{Y/k}^1$ , and thus an inclusion

$$H^0(Y, \Omega_{Y/k}^1(-D)) \subset H^0(Y, \iota_* \iota^* \Omega_{Y/k}^1) = \Omega_{Y/k, \eta}^1.$$

An ample divisor  $D \in \text{Div}(Y)$  is called a *pre-Tango* structure if there is a rational function  $r \in F$  that is not a  $p$ -th power such that the rational differential  $dr \in \Omega_{Y/k, \eta}^1$  comes from a global section  $dr \in H^0(Y, \Omega_{Y/k}^1(-pD))$ . As a short hand, one then writes  $(dr) \geq pD$ . For curves, this notion goes back to Tango [19]. It was used by Raynaud [10] to construct counterexamples for Kodaira Vanishing for surfaces, and was studied in higher dimensions by Mukai [9] and Takeda [17].

The invertible sheaf  $\mathcal{L} = \mathcal{O}_Y(D)$  associated to a pre-Tango structure  $D \in \text{Div}(Y)$  has  $H^1(Y, \mathcal{L}^{\otimes -1}) \neq 0$ , hence is a counterexample to Kodaira Vanishing. Conversely, if  $\mathcal{L}$  is ample with  $H^1(Y, \mathcal{L}^{\otimes -1}) \neq 0$ , then  $\mathcal{L}^{\otimes n}$ , for some integer  $n \geq 1$ , comes from a pre-Tango structure  $D \in \text{Div}(Y)$ , according to [16], Proposition 8.

**Theorem 5.1.** *Assume that  $X$  is another integral smooth projective scheme and let  $\pi : X \rightarrow Y$  be a finite surjective morphism that generically étale. If  $D \in \text{Div}(Y)$  is a pre-Tango structure, then  $\pi^*(D) \in \text{Div}(X)$  is a pre-Tango structure as well.*

*Proof.* Since  $\pi : X \rightarrow Y$  is finite, the preimage  $\pi^*(D)$  of the ample divisor  $D$  remains ample. Since this morphism is also surjective, the two schemes  $X$  and  $Y$  have the

same dimension  $d \geq 0$ . The smoothness of  $X$  and  $Y$  ensures that the coherent sheaves  $\Omega_{X/k}^1$  and  $\Omega_{Y/k}^1$  are locally free of rank  $d$ . Consider the exact sequence

$$\pi^*(\Omega_{Y/k}^1) \longrightarrow \Omega_{X/k}^1 \longrightarrow \Omega_{X/Y}^1 \longrightarrow 0.$$

The term on the right vanishes at the generic point of  $X$ , because the extension of function fields  $k(Y) \subset k(X)$  is separable. Consequently, the kernel  $\mathcal{F}$  for the map  $\pi^*(\Omega_{Y/k}^1) \rightarrow \Omega_{X/k}^1$  on the left has rank zero. Since the scheme  $X$  has no embedded components, the kernel  $\mathcal{F}$  vanishes, and we get an injection  $\pi^*(\Omega_{Y/k}^1) \subset \Omega_{X/k}^1$ . We may regard this as an inclusion inside the quasicoherent sheaf  $\iota_*\iota^*\Omega_{X/k}^1$  of rational differentials, where  $\iota : \text{Spec } k(X) \rightarrow X$  is the inclusion of the generic point. This gives inclusions

$$\pi^*(\Omega_{Y/k}^1(-pD)) \subset \Omega_{X/k}^1(-p\pi^*(D)) \subset \iota_*\iota^*\Omega_{X/k}^1.$$

Since  $D \in \text{Div}(Y)$  is a pre-Tango structure, there is a rational function  $r \in k(Y)$  that is not a  $p$ -th power with  $(dr) \geq pD$ . Using that  $k(Y) \subset k(X)$  is separable, one easily infers that  $r \in k(X)$  is not a  $p$ -th power. The rational differential  $dr \in \Omega_{Y/k,\eta}^1$  extends to a global section  $dr \in H^0(Y, \Omega_{Y/k}^1(-pD))$ . Viewed as element in  $H^0(X, \iota_*\iota^*\Omega_{X/k}^1)$ , it lies in

$$H^0(X, \pi^*(\Omega_{Y/k}^1(-pD))) \subset H^0(X, \Omega_{X/k}^1(-p\pi^*(D))) \subset H^0(X, \iota_*\iota^*\Omega_{X/k}^1).$$

Therefore,  $\pi^*(D) \in \text{Div}(X)$  is a pre-Tango structure.  $\square$

Let us say that  $H^1$ -Kodaira vanishing holds on  $X$  if  $H^1(X, \mathcal{L}^{\otimes -1}) = 0$  for all ample invertible sheaves on  $\mathcal{L}$ . In other words, there are no pre-Tango structures on  $X$ . We record the following consequence:

**Corollary 5.2.** *Assumptions as in the theorem. If  $H^1$ -Kodaira vanishing holds for  $X$ , then it also holds for  $Y$ .*

Finally, we relate our conditions to liftings over the truncated ring of Witt vectors  $W_2(k)$ :

**Corollary 5.3.** *Assume that  $X$  is an integral smooth projective scheme of dimension  $d \geq 2$ , and  $\pi : X \rightarrow Y$  be a finite surjective morphism that generically étale, with  $Y$  smooth. Suppose that  $X$  admits a lifting over  $W_2(k)$ . Then  $H^1$ -Kodaira vanishing holds on  $Y$ , and furthermore  $Y$  is projective.*

*Proof.* According to [4], Proposition 6.6.1, the scheme  $Y$  is projective. The assumption  $d \geq 2$  and the liftability of  $X$  ensures that  $H^1$ -Kodaira vanishing holds for  $X$ , according to Deligne and Illusie ([2], Corollary 2.8). By the previous Corollary,  $H^1$ -Kodaira vanishing holds on  $Y$  as well.  $\square$

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